Lecture(3)

Set Theoretic Operations

Set Theoretic Operations

Set Theoretic Operations

- The basic operations are:
  - Union
  - Intersection
  - Complement

Before introducing these three operations, first we shall define the notion of *containment* which plays a central role in both ordinary and fuzzy sets

Def: Containment or Subset:

Fuzzy set A is contained in B (or,,equivalently A is subset of B or A is smaller than or equal to B) if and only if

$$\mu_A(x) \le \mu_B(x) \quad for \quad all \quad x$$

In symbols:

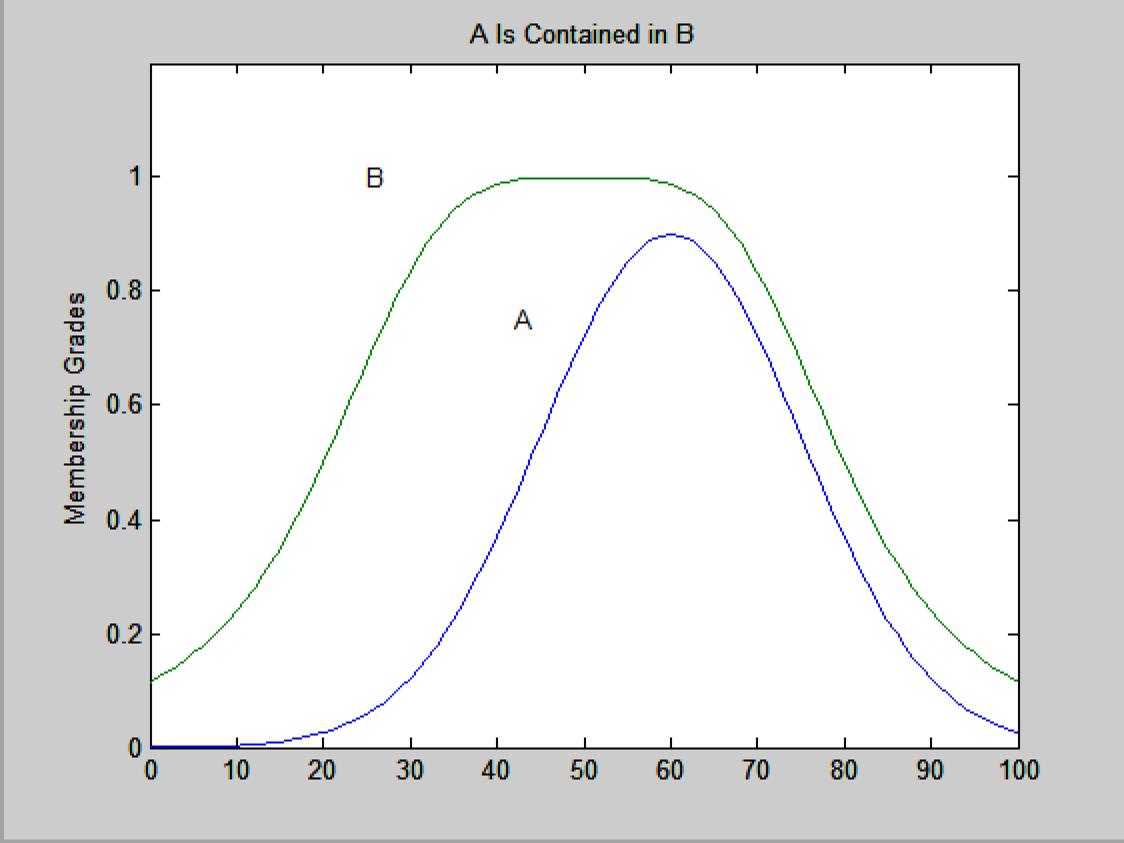
$$A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x)$$

Set Theoretic Operations

illustration oft the Subset concept

$$A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x)$$

```
x = 0:100;
B = gbellmf(x, [30, 2, 50]);
A = 0.9*gaussmf(x, [15, 60]);
plot(x, A, x, B);
axis([-inf inf 0 1.2]);
title('A Is Contained in B');
ylabel('Membership Grades');
%Set (gca, 'xticklabels', []);
text(25, 1.0, 'B');
text(42, 0.75, 'A');
```



Set-Theoretic Operations

Union:(disjunction)

$$C = A \cup B \text{ or } A\ OR\ B \Leftrightarrow \mu_c(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$$

Or alternatively, defined as algebraic sum

$$\mu_{A \cup B} = \mu_A + \mu_B - \mu_A * \mu_B$$

Two kinds of T-conorm

Intersection(or conjunction):

$$C = A \cap B \text{ or } A\ AND\ B \Leftrightarrow \mu_c(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$$

Or alternatively, defined as algebraic product

$$\mu_{A \cap B} = \mu_A * \mu_B$$

Two kinds of T-norm

Complement:

$$\overline{A} = X - A \Leftrightarrow \mu_{\overline{A}}(x) = 1 - \mu_A(x)$$

This corresponds to the logical NOT operation.

Set-Theoretic Operations

Cartesian product and Cartesian co-product:

- Let A and B be fuzzy sets in X and Y, respectively. The ***Cartesian product*** of A and B denoted by $A \times B$ is a fuzzy set in the product space $X \times Y$ with the MF

$$\mu_{A\times B}(x,y)=\min\left(\mu_A(x),\mu_B(y)\right)$$

- Similarly ,the ***Cartesian Co-product*** of A and B denoted by A+B is a fuzzy set with the MF

$$\boxed{\mu_{A+B}(x,y)=\max\left(\mu_A(x),\mu_B(y)\right)}$$

Example [2]

Assume

- A={(1,0), (2,.5),(3,1)}
- B={(1,1), (2,.5),(3,0)}
- $A \times B$ can be arranged as a two-dimensional fuzzy set:

		$\mathcal{B}$		
		1	0.5	0
	0	0	0	0
$\mathcal{A}$	0.5	0.5	0.5	0
	1	1	0.5	0

A × B={((1,1),0), ((1,2),0),((1,3),0) ((2,1),0.5), ((2,2),0.5), ((2,3),0),((3,1),1), ((3,2),0.5) ((3,3),0)}

Example

Example 3.4. Consider two fuzzy sets A and B. A represents universe of three discrete temperatures $x = \{x_1, x_2, x_3\}$ and B represents universe of two discrete flow $y = \{y_1, y_2\}$. Find the fuzzy Cartesian product between them:

$$\underset{\sim}{A} = \frac{0.4}{x_1} + \frac{0.7}{x_2} + \frac{0.1}{x_3} \quad \text{and} \quad \underset{\sim}{B} = \frac{0.5}{\gamma_1} + \frac{0.8}{\gamma_2}.$$

Solution. $\underset{\sim}{A}$ represents column vector of size 3×1 and $\underset{\sim}{B}$ represents column vector of size 1×2. The fuzzy Cartesian product results in a fuzzy relation $\underset{\sim}{R}$ of size 3×2:

$$\underset{\sim}{A} \times \underset{\sim}{B} = \underset{\sim}{R} = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} \gamma_1 & \gamma_2 \end{array} \\ \left[\begin{array}{cc} 0.4 & 0.4 \\ 0.5 & 0.7 \\ 0.1 & 0.1 \end{array} \right] \end{array}.$$

--Set-Theoretic Operations [Rajesekaran ch6]

- Product of a two fuzzy sets
- Equality
- Product of a fuzzy set with a crisp number
- Power of a fuzzy set
- Difference
- Disjunctive sum

Product of a two fuzzy sets

The product of two fuzzy sets A and B is a new set A.B whose MF is defined as

$$\mu_{A\cdot B}(x) = \mu_A(x).\mu_B(x)$$

Example

$$A = \{(x_1, 0.2), (x_2, 0.8), (x_3, 0.4)\}$$

$$B = \{(x_1, 0.4), (x_2, 0), (x_3, 0.1)\}$$

Find A.B

Solution

$$A \cdot B = \{(x_1, 0.08), (x_2, 0), (x_3, 0.04)\}$$

Equality:

two fuzzy sets A and B are said to beequal A=B if

$$\mu_A(x) = \mu_B(x)$$

Example

$$A = \{(x_1, 0.2), (x_2, 0.8)\}$$
$$B = \{(x_1, 0.6), (x_2, 0.8)\}$$
$$C = \{(x_1, 0.2), (x_2, 0.8)\}$$

$$A \neq B$$

$$A = C$$

Product of a fuzzy set with a crisp number

Multiplying a fuzzy set A by a crisp number a results in a new fuzzy set a.A with the MF :

$$\mu_{a\cdot A}(x) = a.\mu_A(x)$$

Example

$$A = \{(x_1, 0.4), (x_2, 0.6), (x_3, 0.8)\}$$

$$a = 0.3$$

$$\Rightarrow a.A = \{(x_1, 0.12), (x_2, 0.18), (x_3, 0.24)\}$$

Power of a fuzzy set

the α power of a fuzzy set A is a new set A^{α} with the MF :

$$\mu_{A^{\alpha}}(x) = (\mu_A(x))^{\alpha}$$

Example

$$A = \{(x_1, 0.4), (x_2, 0.6), (x_3, 0.8)\}$$

$$a = 2$$

$$A^{\alpha} = \{(x_1, 0.16), (x_2, 0.36), (x_3, 0.64)\}$$

Difference

Difference of two fuzzy sets A and B is a new set A-B defined as:

$$A - B = A \cap \overline{B}$$

Example

$$A = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.6)\}$$

$$B = \{(x_1, 0.1), (x_2, 0.4), (x_3, 0.5)\}$$

Find A-B

Solution

$$\overline{B} = \{(x_1, 0.9), (x_2, 0.6), (x_3, 0.5)\}$$

$$A - B = A \cap \overline{B} = \{(x_1, 0.2), (x_2, 0.5), (x_3, 0.5)\}$$

Disjunctive sum

The disjunctive sum of two fuzzy sets A and B is a new set $A \oplus B$ defined as:

$$A \oplus B = (\overline{A} \cap B) \cup (A \cap \overline{B})$$

Example

$$A = \{(x_1, 0.4), (x_2, 0.8), (x_3, 0.6)\} \quad B = \{(x_1, 0.2), (x_2, 0.6), (x_3, 0.9)\}$$

Find $A \oplus B$

Solution

$$\overline{A} = \{(x_1, 0.6), (x_2, 0.2), (x_3, 0.4)\} \quad \overline{B} = \{(x_1, 0.8), (x_2, 0.4), (x_3, 0.1)\}$$

$$\overline{A} \cap B = \{(x_1, 0.2), (x_2, 0.2), (x_3, 0.4)\} \quad A \cap \overline{B} = \{(x_1, 0.4), (x_2, 0.4), (x_3, 0.1)\}$$

$$\boxed{A \oplus B = (\overline{A} \cap B) \cup (A \cap \overline{B}) = (x_1, 0.4), (x_2, 0.4), (x_3, 0.4)}$$

Properties of the fuzzy sets

- The properties of the classical set also suits for the properties of the fuzzy sets. The important properties of fuzzy set includes:
- Commutativity

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

- Associativity

$$A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributivity

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Properties of the fuzzy sets

- Idempotency

$$A \cup A = A, \qquad A \cap A = A.$$

- Identity

$$A \cup \varphi = A \quad \text{and } A \cap X = A \;, A \cap \varphi = \varphi \text{ and } A \cup X = X.$$

where

Φ is the empty set (the degree of membership of all its elements is zero)

- Transitivity

If $A \subset B \subset C$ then $A \subset C$

- Involution (double complement)

.

$$\bar{\bar{A}} = A$$

Set Theoretic Operations

Example

Consider the universe U and the two sets A(u) and B(u) in the following table .Determine :

$$A \cup B$$

$$A \cap B$$

$$\overline{A}$$

U	0	1	2	3	4	5	6	7	8	9	comment
A(u)	0	0	0.1	0.2	0.3	0.8	0.9	1	1	1	
B(u)	1	1	0.9	0.8	0.7	0.5	0.4	0.2	0.2	0	
$A \cup B$	1	1	0.9	0.8	0.7	0.8	0.9	1	1	1	Max(A,B)
$A \cap B$	0	0	0.1	0.2	0.3	0.5	0.4	0.2	0.2	0	min(A,B)
$\overline{A}$	1	1	0.9	0.8	0.7	0.2	0.1	0	0	0	1-A

Example

Consider two fuzzy sets *A* and *B. F*ind Complement, Union, Intersection

$$\underset{\sim}{A} = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.6}{4} + \frac{0.2}{5} + \frac{0.6}{6} \right\},$$

$$\underset{\sim}{B} = \left\{ \frac{0.5}{2} + \frac{0.8}{3} + \frac{0.4}{4} + \frac{0.7}{5} + \frac{0.3}{6} \right\}.$$

Solution

Complement

$$\bar{\underset{\sim}{A}} = \left\{ \frac{0}{2} + \frac{0.5}{3} + \frac{0.4}{4} + \frac{0.8}{5} + \frac{0.4}{6} \right\},$$

$$\bar{\underset{\sim}{B}} = \left\{ \frac{0.5}{2} + \frac{0.2}{3} + \frac{0.6}{4} + \frac{0.3}{5} + \frac{0.7}{6} \right\}.$$

Union

$$\underset{\sim}{A} \cup \underset{\sim}{B} = \left\{ \frac{1}{2} + \frac{0.8}{3} + \frac{0.6}{4} + \frac{0.7}{5} + \frac{0.6}{6} \right\}.$$

Max is used

…….Solution

$$\underset{\sim}{A} = \left\{ \frac{1}{2} + \frac{0.5}{3} + \frac{0.6}{4} + \frac{0.2}{5} + \frac{0.6}{6} \right\},$$
$$\underset{\sim}{B} = \left\{ \frac{0.5}{2} + \frac{0.8}{3} + \frac{0.4}{4} + \frac{0.7}{5} + \frac{0.3}{6} \right\}.$$

intersection

$$\underset{\sim}{A} \cap \underset{\sim}{B} = \left\{ \frac{0.5}{2} + \frac{0.5}{3} + \frac{0.4}{4} + \frac{0.2}{5} + \frac{0.3}{6} \right\}.$$

min is used

Lecture(4)

- Fuzzy Relations

Topics to be covered

- Extension principle of fuzzy sets
- Fuzzy relations
- Projection of fuzzy relations
- Cylindrical extension of fuzzy relations
- Fuzzy max-min and max-product composition operation

Extension principle

- The extension principle is one of the fundamentals of fuzzy sets theory
- It provides a general procedure for extending crisp domains of mathematical expression to fuzzy domain.
- وهو يوفر إجراء عام لتوسيع مجالات واضحة للتعبير الرياضي لنطاق غامض.
- It generalizes the common point to pint mapping of a function f(.) to a mapping between fuzzy sets
  - فإنه يعمم نقطة مشتركة لإشارة رسم خرائط الدالة f(.) للتعيين بين المجموعات الضبابية.
- It represent a mean to get a fuzzy model for a variable if we know the fuzzy model for another variable and the functional relationship between them.
  - انها تمثل وسيلة للحصول على نموذج غامض لمتغير إذا علمنا نموذج غامض لمتغير آخر، والعلاقة الوظيفية بينهما.
- *For example, we know that the voltage across a resistor is connected to the current with the formula: I = V/R . We want to know how our information about the voltage, expressed as a fuzzy set, can be used to get a model for the current.*

What do we need this extension principle for?

- It gives us the rule of how to calculate an output of a fuzzy system
- If we know the structure of the system, containing algebraic and logical blocks, and the system inputs are fuzzy, on the basis of this principle we can determine the outputs of the system..

Extension Principle

Suppose that

***A* is a fuzzy set on *X* :**

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n$$

The image of *A* under *f( )* is a fuzzy set *B*:

$$\boxed{\mathbf{B = f(A) = \mu_A(x_1)/y_1 + \mu_A(x_2)/y_2 + \cdots + \mu_A(x_n)/y_n}}$$

where $y_i = f(x_i)$, $i = 1$ to n.

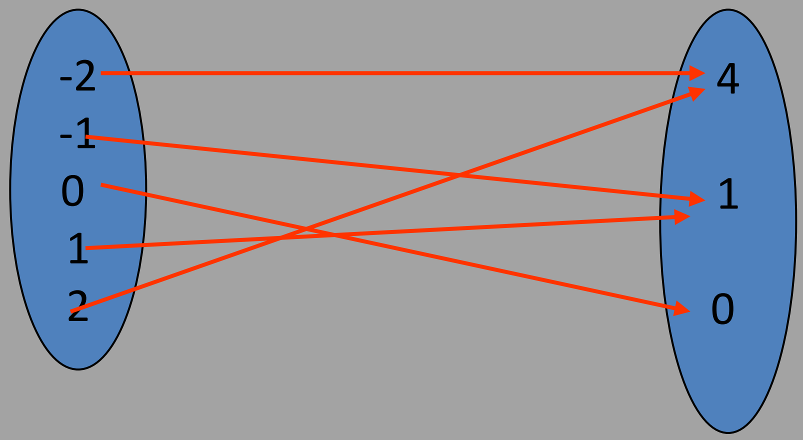
If *f( )* is a many-to-one mapping, then

$$\mu_B(y) = \max_{x = f^{-1}(y)} \mu_A(x)$$

Example: Application of the extension principle to fuzzy sets with discrete UOD

Let $A = 0.3/-2 + 0.4/-1 + 0.8/0 + 1/1 + 0.7/2\}$

and $y = f(u) = u^2$. *(many to one mapping)*



Then

$B = 0.3/4 + 0.4/1 + 0.8/0 + 1/1,\ 0.7/4$

$= (0.3 \vee 0.7)/4 + (0.4 \vee 1)/1 + 0.8/0$

$= 0.7/4 + 1/1 + 0.8/0$

Crisp relation

Crisp relation determine the relation between the elements of two crisp sets and is defined by

$$\mu_R(x,y)=\begin{Bmatrix} 1 & (x,y)\in X\times Y \\ 0 & (x,y)\notin X\times Y \end{Bmatrix}$$

Where:

1 means complete relation and

0 means no relation

Ex:

X={1,2}   Y={A,B}

The Cartesian space is $X\times Y=\{(1,A),(1,B),(2,A),(2,B)\}$

When the sets are finite the relation is represented by a matrix R called the ***relation matrix***

----Crisp relation

For the sets X,Y a relation may be:

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

And this means that:

- The elements 1 and A are not related
- The elements 1 and B are related
- The elements 2 and A are related
- The elements 2 and B are not related

Composition of crisp relation

- X, Y, Z are universal sets
- Let R be a relation between X and Y
- Let S be a relation between Y and Z
- Let T be a relation between X and Z
- If we know R and S. can we find T?
- The answer is yes: and it can be done using a knowledge inference methods known as composition

Composition of crisp relation

$$T = R \circ S$$

The best known compositing operations are:

Max-min composition:

$$\boxed{\mu_T = \vee_y [\mu_R(x,y) \wedge \mu_S(x,y)}$$

Max- product

$$\boxed{\mu_T = \vee_y [\mu_R(x,y)\mu_S(x,y)}$$

Example

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Using ***Max-min composition:***

$$T(1,1) = \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{row1\ of\ R} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{C1\ of\ S} \xrightarrow{\min} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{\max} 1$$

$$\boxed{T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

Using ***Max-product composition:***

$$T(1,1) = \underset{row1\ of\ R}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \underset{C1\ of\ S}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \xrightarrow{product} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \xrightarrow{\max} 1$$

$$\boxed{T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

Comment: for crisp relation the two composition methods give the same result

Composition example

Given

R=

	y_1	y_2	y_3	y_4
x_1	1	0	1	0
x_2	0	0	0	1
x_3	0	0	0	0

S=

	z_1	z_2
y_1	0	1
y_2	0	0
y_3	0	1
y_4	0	0

Find $T = R \circ S$

Answer

T=

	z_1	z_2
x_1	0	1
x_2	0	0
x_3	0	0

Fuzzy Relations

- A fuzzy relation *R* is a 2D MF:

$$R = \{((x, y), \mu_R(x, y)) | (x, y) \in X \times Y\}$$

- Examples:
  - x is close to y (x and y are numbers)
  - x depends on y (x and y are events)
  - x and y look alike (x, and y are persons or objects)
  - If x is large, then y is small (x is an observed reading and Y is a corresponding action)*

Ex 3.3 [6]

- Let X=Y=R^+ (+ive real line)and R=“ y is much greater than x” . The mf of the fuzzy relation can be subjectively defined as

$$\mu_R(x,y)=\begin{cases} \dfrac{y-x}{x+y+2} & if \quad y \succ x \\ 0 \quad if & y \le x \end{cases}$$

- **If X={3,4,5} and Y={3,4,5,6,7} then the fuzzy relation R in a matrix format is:**

$$R=\begin{bmatrix} 0 & 0.111 & 0.2 & 0.237 & 0.333 \\ 0 & 0 & 0.091 & 0.167 & 0.231 \\ 0 & 0 & 0 & 0.077 & 0.143 \end{bmatrix}$$

Projection of fuzzy relations

- For a fuzzy relation R(x,y), the projection on X denoted by R1 is given by:

$$\mu_{R1}(x) = \max_{y}(\mu_R(x,y))$$

Example

$$R = \begin{bmatrix} 0 & 0.111 & 0.2 & 0.237 & 0.333 \\ 0 & 0 & 0.091 & 0.167 & 0.231 \\ 0 & 0 & 0 & 0.077 & 0.143 \end{bmatrix} \xrightarrow{\text{X projection}} R_1 = \begin{bmatrix} 0.333 \\ 0.231 \\ 0.143 \end{bmatrix}$$

- **Similarly the projection on Y denoted by R2 is given by:**

$$\mu_{R2}(y) = \max_{x}(\mu_R(x,y))$$

Example

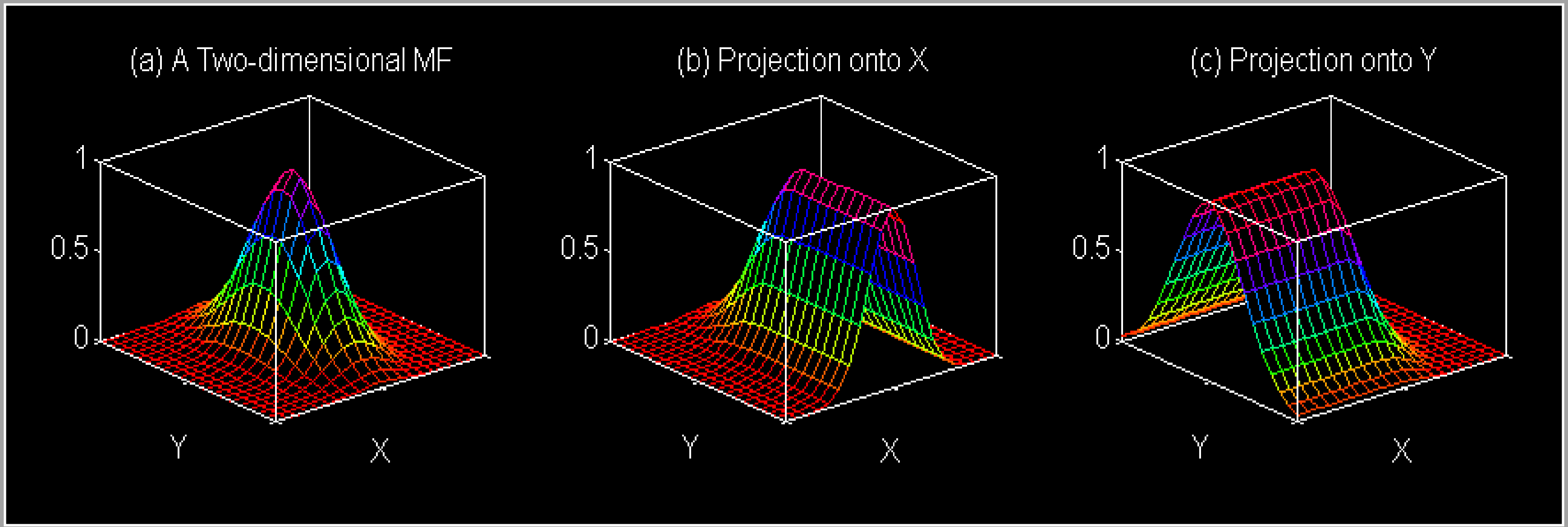
$$R = \begin{bmatrix} 0 & 0.111 & 0.2 & 0.237 & 0.333 \\ 0 & 0 & 0.091 & 0.167 & 0.231 \\ 0 & 0 & 0 & 0.077 & 0.143 \end{bmatrix}$$

$$\Downarrow$$

$$R2 = \begin{bmatrix} 0 & 0.111 & 0.2 & 0.237 & 0.333 \end{bmatrix}$$

2D MF Projection

Two-dimensional MF	Projection onto X	Projection onto Y



$\mu_R(x, y)$

$\mu_A(x) = \max_y \mu_R(x, y)$

$\mu_B(y) = \max_x \mu_R(x, y)$

project.m

Cylindrical extension of fuzzy relations

- Cylindrical extension from an x projection is done by filling all the columns of the matrix by the x projection

$$R = \begin{bmatrix} 0 & 0.111 & 0.2 & 0.237 & 0.333 \\ 0 & 0 & 0.091 & 0.167 & 0.231 \\ 0 & 0 & 0 & 0.077 & 0.143 \end{bmatrix} \xrightarrow{\text{X projection}} R1 = \begin{bmatrix} 0.333 \\ 0.231 \\ 0.143 \end{bmatrix} \xrightarrow{\text{Cylindrical ext}} R = \begin{bmatrix} 0.333 & 0.333 & 0.333 & 0.333 & 0.333 \\ 0.231 & 0.231 & 0.231 & 0.231 & 0.231 \\ 0.143 & 0.143 & 0.143 & 0.143 & 0.143 \end{bmatrix}$$

- **Cylindrical extension from a Y projection is done by filling all the rows of the matrix by the Y projection**

$$R = \begin{bmatrix} 0 & 0.111 & 0.2 & 0.237 & 0.333 \\ 0 & 0 & 0.091 & 0.167 & 0.231 \\ 0 & 0 & 0 & 0.077 & 0.143 \end{bmatrix}$$

$$\downarrow$$

$$R2 = \begin{bmatrix} 0 & 0.111 & 0.2 & 0.237 & 0.333 \end{bmatrix}$$

$$R2 = \begin{bmatrix} 0 & 0.111 & 0.2 & 0.237 & 0.333 \end{bmatrix}$$

$$\uparrow$$

$$R = \begin{bmatrix} 0 & 0.111 & 0.2 & 0.237 & 0.333 \\ 0 & 0.111 & 0.2 & 0.237 & 0.333 \\ 0 & 0.111 & 0.2 & 0.237 & 0.333 \end{bmatrix}$$

Composition

- Fuzzy relations in different product spaces can be combined through compositing operation to infer a new relation (knowledge inference)
- The best known compositing operations are:
  - ***Max-min composition***
  - ***Max- product***
- The max-min composition of two fuzzy relations *R1* (defined on *X* and *Y*) and *R2* (defined on *Y* and *Z*) is

$$\mu_{R_1 \circ R_2}(x,z) = \vee_y [\mu_{R_1}(x,y) \wedge \mu_{R_2}(y,z)]$$

Properties:

- Associativity:

$$R \circ (S \circ T) = (R \circ S) \circ T$$

- Distributivity over union:

$$R \circ (S \bigcup T) = (R \circ S) \bigcup (R \circ T)$$

- Week distributivity over intersection:

$$R \circ (S \bigcap T) \subseteq (R \circ S) \bigcap (R \circ T)$$

- Monotonicity:

$$S \subseteq T \Rightarrow (R \circ S) \subseteq (R \circ T)$$

Example 1:

let R1=‘ x is relevant to y’’

let R2=‘ y is relevant to z’’

Find the relation R3= ‘’x is relevant to z’’ using max-min composition

$$\boldsymbol{R1} = \boldsymbol{R2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

solution

$$\mu_{R1\circ R2}(x1, z1) = \max(\min(0,0), \min(1,1)) = \max(0,1) = 1$$ Row1,col1

$$\mu_{R1\circ R2}(x1, z2) = \max(\min(0,1), \min(1,0)) = \max(0,0) = 0$$ Row1,col2

$$\mu_{R1\circ R2}(x2, z1) = \max(\min(1,0), \min(0,1)) = \max(0,0) = 0$$ Row2,col1

$$\mu_{R1\circ R2}(x2, z2) = \max(\min(1,1), \min(0,0)) = \max(1,0) = 1$$ Row2,col2

$$\boldsymbol{R3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 2

Using max–min composition find relation between R and S:

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}, \quad S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array}.$$

Solution. The max–min composition is given by:

$$\begin{aligned}
\mu_T(x_1, z_1) &= \max\,(\min\,(1,\,0),\, \min\,(1,\,1),\, \min\,(0,\,1)) \\
&= \max\,[0,\,1,\,0] = 1, \\
\mu_T(x_1, z_2) &= \max\,(\min\,(1,\,1),\, \min\,(1,\,0),\, \min\,(1,\,1)) \\
&= \max\,[1,\,0,\,1] = 1, \\
\mu_T(x_2, z_1) &= \max\,(\min\,(0,\,0),\, \min\,(0,\,1),\, \min\,(1,\,1)) \\
&= \max\,[0,\,0,\,1] = 1, \\
\mu_T(x_2, z_2) &= \max\,(\min\,(0,\,1),\, \min\,(0,\,0),\, \min\,(1,\,1)) \\
&= \max\,[0,\,0,\,1] = 1, \\
\mu_T(x_3, z_1) &= \max\,(\min\,(0,\,0),\, \min\,(1,\,1),\, \min\,(0,\,1)) \\
&= \max\,[0,\,1,\,0] = 1, \\
\mu_T(x_3, z_2) &= \max\,(\min\,(0,\,1),\, \min\,(1,\,0),\, \min\,(0,\,1)) \\
&= \max\,[0,\,0,\,0] = 0,
\end{aligned}$$

$$R \circ S = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Example 3:

let R1=‘ x is relevant to y’’. let R2=‘ y is relevant to z’’

Find the relation R3= ‘’x is relevant to z’’ using max-min composition

$$R1 = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix} \qquad R2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix}$$

solution

From Row2 of R1 ,col1 of R2

$$\mu_{R1 \circ R2}(x2, z1) = \max(0.4 \wedge 0.9, 0.2 \wedge 0.2, 0.8 \wedge 0.5, 0.9 \wedge 0.7) =$$
$$\max(0.4, 0.2, 0.5, 0.7) = 0.7$$

Similarly we can find the other entries of R3

R3 =

0.7000	0.5000
0.7000	0.6000
0.6000	0.3000

function : max_star.m

Max-product Composition

- Max-product composition:

$$\mu_{R_1 \circ R_2}(x,z) = \vee_y [\mu_{R_1}(x,y)\mu_{R_2}(y,z)]$$

Example 4:

let R1=‘ x is relevant to y’’. let R2=‘ y is relevant to z’’

Find the relation R3= ‘’x is relevant to z’’ using **max-product** composition

$$R1 = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix} \qquad R2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix}$$

solution

From Row2 of R1 ,col1 of R2

$$\mu_{R1 \circ R2}(x2, z1) = \max(0.4 \times 0.9, 0.2 \times 0.2, 0.8 \times 0.5, 0.9 \times 0.7) =$$
$$\max(0.36, 0.04, 0.4, 0.63) = 0.63$$

Similarly we can find the other entries of R3

R3 =

0.4900	**0.3000**
0.6300	**0.4800**
0.5400	**0.2400**

function : max_star.m

Example 3.5. Consider fuzzy relations:

$$\underset{\sim}{R} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.7 & 0.6 \\ 0.8 & 0.3 \end{bmatrix} \end{array}, \quad \underset{\sim}{S} = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_2 \end{array} \\ \begin{bmatrix} 0.8 & 0.5 & 0.4 \\ 0.1 & 0.6 & 0.7 \end{bmatrix} \end{array}.$$

Find the relation $T = \underset{\sim}{R} \circ \underset{\sim}{S}$ using max–min and max–product composition.

Solution. **Max–Min Composition**

$$
\begin{aligned}
T &= \underset{\sim}{R} \circ \underset{\sim}{S} \\
\mu_T(x_1, z_1) &= \max\left[\min(0.7, 0.8), \min(0.6, 0.1)\right] \\
&= \max\left[0.7, 0.1\right] \\
&= 0.7, \\
\mu_T(x_1, z_2) &= \max\left[\min(0.7, 0.5), \min(0.6, 0.6)\right] \\
&= \max\left[0.5, 0.6\right] \\
&= 0.6, \\
\mu_T(x_1, z_3) &= \max\left[\min(0.7, 0.4), \min(0.6, 0.7)\right] \\
&= \max\left[0.4, 0.7\right] \\
&= 0.7, \\
\mu_T(x_2, z_1) &= \max\left[\min(0.8, 0.8), \min(0.3, 0.1)\right] \\
&= \max\left[0.8, 0.1\right] \\
&= 0.8, \\
\mu_T(x_2, z_2) &= \max\left[\min(0.8, 0.5), \min(0.3, 0.6)\right] \\
&= \max\left[0.5, 0.3\right] \\
&= 0.5, \\
\mu_T(x_2, z_3) &= \max\left[\min(0.8, 0.4), \min(0.3, 0.7)\right] \\
&= 0.4,
\end{aligned}
$$

$$
\underset{\sim}{S} = \begin{array}{c} \\ x_1 \\ x_2 \end{array}
\begin{array}{c}
\begin{array}{ccc} z_1 & z_2 & z_2 \end{array} \\
\begin{bmatrix} 0.7 & 0.6 & 0.7 \\ 0.8 & 0.5 & 0.4 \end{bmatrix}
\end{array}.
$$

Max–Product Composition

$$
\begin{aligned}
\mu_T(x_1, z_1) &= \max[\min(0.7 \times 0.8), \min(0.6 \times 0.1)] \\
&= \max[0.56, 0.06] \\
&= 0.56, \\
\mu_T(x_1, z_2) &= \max[\min(0.7 \times 0.5), \min(0.6 \times 0.6)] \\
&= \max[0.35, 0.36] \\
&= 0.36, \\
\mu_T(x_1, z_3) &= \max[\min(0.7 \times 0.4), \min(0.6 \times 0.7)] \\
&= \max[0.28, 0.35] \\
&= 0.35,
\end{aligned}
$$

$$
\begin{aligned}
\mu_T(x_2, z_1) &= \max[\min(0.8 \times 0.8), \min(0.3 \times 0.1)] \\
&= \max[0.64, 0.03] \\
&= 0.64, \\
\mu_T(x_2, z_2) &= \max[\min(0.8 \times 0.5), \min(0.3 \times 0.6)] \\
&= \max[0.40, 0.18] \\
&= 0.40, \\
\mu_T(x_2, z_3) &= \max[\min(0.8 \times 0.4), \min(0.3 \times 0.7)] \\
&= \max[0.32, 0.21] \\
&= 0.32,
\end{aligned}
$$

$$
\underset{\sim}{T} = \begin{bmatrix} 0.56 & 0.36 & 0.42 \\ 0.64 & 0.40 & 0.32 \end{bmatrix}.
$$